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Abstract: This paper is about the properties of Markov-switching rational expectations (MSRE) models. We discuss possible solution concepts for MSRE models, distinguishing between stationary and bounded equilibria. For the case of models with one variable, we provide a necessary and sufficient condition for uniqueness of a bounded equilibrium, and we relate this condition to an alternative, the generalized Taylor principle suggested by Davig and Leeper. We provide examples of models with multiple bounded and multiple stationary equilibria which suggest that it may be more difficult to rule out nonfundamental equilibria in MSRE models than in the single-regime case where the Taylor principle is known to guarantee local uniqueness.

JEL classification: E5

Key words: policy rule, inflation, serial dependence, multiple equilibria, stationary equilibrium, bounded equilibrium

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INDETERMINACY IN A FORWARD LOOKING REGIME SWITCHING MODEL

I. INTRODUCTION

Work by Richard Clarida, Jordi Galí and Mark Gertler (2000a) has stimulated recent interest in models where monetary policy may occasionally change between a passive regime in which there exists an indeterminate set of sunspot equilibria and an active regime in which equilibrium is unique. When the policy changes are modeled using a Markov process, the resulting model is inherently non-linear.

Papers in the literature on nonlinear rational expectations models typically compute a solution to functional equations using numerical methods, but not much is known about the analytical properties of these equations. Obtaining a complete set of indeterminate equilibria even for a simple MSRE model is a very difficult problem, and to the best of our knowledge there are no systematic methods to accomplish this task.

The distinction between the linear rational expectations (RE) model and the MSRE model is subtle but important and the conditions for existence and boundedness of a unique solution are different in the two cases. In this paper we study a simple Markov-switching model of inflation that combines two purely forward-looking rational expectations models. The first one has a unique determinate equilibrium and the second is associated with a set of indeterminate equilibria. The MSRE model switches between the two models with transition probabilities governed by a Markov chain.

Within the MSRE environment we first establish conditions for the existence of a minimum-state-variable equilibrium. We go on to discuss alternative definitions of stationarity for non-linear models and we argue that mean-square stability is an appropriate and appealing concept. We then show through a series of examples, based on a one equation model, that there exists a set of mean-square-stable indeterminate equilibria for large open sets of the model's parameter values. Our results imply that the existence of stationary and bounded indeterminate equilibria in MRSE models is a pervasive phenomenon that cannot be ruled out in all regimes by the actions of the policy maker in a single regime. The examples in the current paper demonstrate that a uniqueness criterion proposed by Davig and Leeper (2007), the "generalized Taylor principle", holds in the one equation model if and only if the coefficient on the government reaction function is restricted to be positive. In a companion paper, Farmer, Waggoner, and Zha (2007), we show that the principle fails more generally in multi-equation models.

II. The Model

Following Robert King (2000) and Michael Woodford (2003), we study a simple flexible price model in which the central bank can affect inflation but not the real interest rate. In this model, the Fisher equation links the real interest rate, r_t, and the nominal interest rate, R_t, by the equation,

$$R_t = E_t\left[\pi_{t+1}\right] + r_t, \tag{1}$$

where E_t is the conditional expectation at date t and π_{t+1} is the inflation rate at date $t+1$. The central bank sets the time-varying rule

$$R_t = \phi_{\xi_t}\pi_t - \kappa_{\xi_t}\varepsilon_t, \tag{2}$$

where ξ_t is a two-state Markov process with transition probabilities $(p_{i,j})$ and $p_{i,j}$ is the probability of transiting from state j to state i. The stochastic process $\{\varepsilon_t\}_{t=1}^{\infty}$ is independently and identically distributed with zero mean and is independent of $\{\xi_t\}_{t=1}^{\infty}$. Substituting Eq (2) into Eq (1) gives the following forward-looking inflation process

$$\phi_{\xi_t}\pi_t = E_t\left[\pi_{t+1}\right] + r_t + \kappa_{\xi_t}\varepsilon_t. \tag{3}$$

We assume that the real interest rate evolves exogenously according to

$$r_t = \rho r_{t-1} + \nu_t, \tag{4}$$

where $|\rho| < 1$ and $\{\nu_t\}_{t=2}^{\infty}$ is independently and identically distributed with zero mean and is independent of $\{\xi_t\}_{t=1}^{\infty}$ and $\{\varepsilon_t\}_{t=1}^{\infty}$. The distributions of both ε_t and ν_t are assumed to have a bounded support.

III. Understanding Indeterminate Equilibrium

In this section we study a special case of the model for which $\phi_{\xi_t} = \phi$ and $\kappa_{\xi_t} = \kappa$ for all ξ_t and $r_t = 0$ for all t. This single regime, or constant-parameter, case helps understand key characteristics of indeterminate equilibria and provides the economic intuition that explains how indeterminate solutions in a completely *forward-looking* model depend on past states, a phenomenon that is not commonly understood. For this special case, Eq (3) can be written as,

$$\pi_t = \frac{1}{\phi} E_t[\pi_{t+1}] + \frac{\kappa}{\phi}\varepsilon_t. \tag{5}$$

Before answering the question of whether or not there is a unique solution of (3), one must decide the properties that a solution will be required to have. At the very least, one wishes to rule out explosive behavior. This can be accomplished in a variety of ways, the most common of which are requiring solutions to be bounded in mean, requiring solutions to be stationary, or requiring solutions to be bounded. Fortunately, in the constant parameter case these are equivalent; that is, there will be a unique solution that is bounded in mean if and only if there is a unique solution that is stationary and if and only if there is a unique solution that is bounded. However, this is not the case when there are multiple regimes. We will return to this phenomena in subsequent sections. In anticipation of later sections, we will assume that solutions are stationary. Since stationary processes are also bounded in mean, this implies there is an $M > 0$ such that $|E_t \pi_{t+s}| < M$ for all s and t.

How might one establish that there is only one stationary solution to the model? Iterating Eq (5) forward for T periods leads to the following expression:

$$\pi_t = \frac{\kappa}{\phi}\varepsilon_t + \left(\frac{1}{\phi}\right)^T E_t \pi_{t+T}. \tag{6}$$

When $T = 1$, this is simply Eq (5). Eq (6) can be shown to hold for all T via an induction argument using the fact that $E_t \pi_{t+T} = \frac{1}{\phi} E_t \pi_{t+T+1}$. Thus

$$\pi_t = \frac{\kappa}{\phi}\varepsilon_t + \lim_{T\to\infty} \left(\frac{1}{\phi}\right)^T E_t \pi_{t+T}. \tag{7}$$

Since $E_t \pi_{t+T}$ is bounded, if $|\phi| > 1$ it follows that

$$\pi_t = \frac{\kappa}{\phi}\varepsilon_t, \tag{8}$$

is the only stationary solution. Following McCallum (1983), a the solution of Eq (5) with this form is referred to as a *minimal state variable* (MSV) solution.

If $|\phi| < 1$ then Eq (8) is still a solution to Eq (5) but there are now additional solutions of the form[1]

$$\pi_t = \frac{\kappa}{\phi}\varepsilon_t + w_t, \tag{9}$$

$$w_t = \phi w_{t-1} + (m\varepsilon_t + \gamma_t), \tag{10}$$

where m is any real number and $\{\gamma_t\}_{t=2}^{\infty}$ is any bounded stochastic process with zero mean that is independent of $\{\varepsilon_t\}_{t=1}^{\infty}$. Solutions in this class are referred to as *sunspot equilibria* following Cass and Shell (1983) who coined the term to refer to equilibria in which the allocations in general equilibrium models may differ across states even when all fundamentals are the same. Because $\phi < 1$, these solutions will be stationary.

Why should we be interested in stationary sunspot solutions of the kind represented by Eq (9)? These solutions have an important feature that qualitatively differs from the MSV solution (8): π_t is serially dependent. Thus either fundamental or sunspot shocks will be propagated through this serial dependence; the inflation process will be more persistent and volatile under the passive policy ($|\phi| < 1$) than that under the active policy ($|\phi| > 1$). Therefore, in the indeterminate equilibrium monetary policy tends to destabilize the inflation process, a point made by Clarida, Galí, and Gertler (2000b), Lubik and Schorfheide (2004),and Boivin and Giannoni (2006).

IV. An Appropriate Equilibrium Concept

Much of the previous work on dynamic stochastic general equilibrium theory has been concerned with constant parameter models. A typical way to proceed is to specify preferences, technology and endowments and to make an explicit assumption about the nature of the stochastic shocks. Sometimes it is possible to specify an environment in which, in the absence of shocks, there exists a unique stationary perfect foresight equilibrium. An example is the single agent real business cycle model. When the stationary equilibrium is unique it may be possible to approximate the stochastic rational expectations equilibrium by linearizing around the steady state and solving for an approximate stochastic rational expectations equilibrium. For this approximation to be reasonable, the stochastic shocks must be small. This is necessary to keep the system close to the non-stochastic steady state, which is the only point in the state space for which the linear approximation is exact.

The non-stochastic dynamics of a perfect foresight linear model are completely characterized by the roots of the characteristic polynomial of a first order matrix difference equation. When all of these roots lie within the unit circle, the stochastic process is stationary and, as a consequence, it is possible to prove theorems which assert that as the variance of the shocks goes to zero, the approximation error vanishes.

[1] An alternative way to express these multiple solutions is

$$\pi_t = \phi\pi_{t-1} - \kappa\varepsilon_{t-1} + (m + \frac{\kappa}{\phi})\varepsilon_t + \gamma_t,$$

which is in the same form as in Lubik and Schorfheide (2004).

One would like to prove a similar theorem for the Markov switching model but since the model is inherently *non-linear*, this is impossible. To make progress with models of this kind one needs an appropriate equilibrium concept different from that used for linear RE models. Specifically, we will describe two different concepts: stationary and bounded solutions. By a stationary solution we mean a process that is mean-square-stable, a concept that we will define below, and by a bounded solution we mean one for which all sample paths remain bounded.[2]

V. Determinate and Indeterminate Solutions and the Taylor Principle

In single regime models there is a simple test for uniqueness that involves counting unstable roots and nonpredetermined variables.[3] Rational expectations equilibrium is unique if the number of non-predetermined variables is equal to the number of unstable roots. This root counting condition lies behind the Taylor principle; that monetary equilibrium will be locally unique if the central bank follows a monetary policy in which it raises the interest rate in response to inflation with a response coefficient greater than one in absolute value.

In a related paper, Davig and Leeper (2007) have tackled the question of how to think about indeterminacy in models of regime switching. Their idea is to find a condition, similar to the Taylor principle, that applies to MSRE models. Davig and Leeper define determinacy to mean the existence of a unique bounded solution to an expanded stochastic linear RE system as

$$\begin{bmatrix}\phi_1 & 0\\ 0 & \phi_2\end{bmatrix}\begin{bmatrix}\pi_{1\,t}\\ \pi_{2\,t}\end{bmatrix} = \begin{bmatrix}p_{1,1} & p_{2,1}\\ p_{1,2} & p_{2,2}\end{bmatrix}\begin{bmatrix}E_t\pi_{1\,t+1}\\ E_t\pi_{2\,t+1}\end{bmatrix} + \begin{bmatrix}r_t + \kappa_{\xi_t}\varepsilon_t\\ r_t + \kappa_{\xi_t}\varepsilon_t\end{bmatrix}, \tag{11}$$

where $\pi_{1\,t}$ and $\pi_{2\,t}$ are newly introduced random variables that follow the *linear* stochastic process (11). By imposing the restriction that the interest rate coefficient of the Taylor rule must be positive ($\phi_i > 0$), Davig and Leeper (2007) correctly show that the necessary and sufficient condition for the *linear* model (11) to have a unique bounded solution for $\pi_{1\,t}$ and $\pi_{2\,t}$ is

Condition 1. All the eigenvalues of the matrix

$$\begin{bmatrix}\phi_1^{-1} & 0\\ 0 & \phi_2^{-1}\end{bmatrix}\begin{bmatrix}p_{1,1} & p_{2,1}\\ p_{1,2} & p_{2,2}\end{bmatrix} \tag{12}$$

lie inside the unit circle.[4]

[2]The reader is referred to Costa, Fragoso, and Marques (2004) for a discussion of mean-square-stability.

[3]As Sims (2001) points out, this test does not always work and the exact condition is more complicated.

[4]Note that our transition matrix P is a transpose of the transition matrix in Davig and Leeper (2007).

It is a well known fact from the standard results for linear RE models that Condition 1 is valid no matter whether ϕ_i is negative or not (see, for example, Blanchard and Kahn (1980), King and Watson (1998), and Sims (2002)). Thus, the restriction $\phi_i > 0$ is unnecessary for the "generalized Taylor principle" stated in Condition 1 to hold.

By letting $\pi_t = \pi_{it}$ when $\xi_t = i$ for $i = 1, 2$, Davig and Leeper (2007) claim that such a uniqueness condition applies to the original MSRE model (3) and call this condition a 'long-run Taylor principle' or 'generalized Taylor principle.' If Condition 1 holds there exists a unique bounded equilibrium for the *nonlinear* model (3). If it fails there may be multiple equilibria driven by non-fundamental shocks.

We have two criticisms of the Davig-Leeper result. First, we think the positivity restriction is not merely for mathematical convenience, but rather an indication that there does not exist in general an equivalence between the existence of a unique bounded equilibrium for a MSRE model and the generalized Taylor principle derived from the linear RE counterpart.[5] Mathematically, since it follows from the standard results that Condition 1 is valid no matter whether ϕ_i is negative or not, Davig and Leeper's generalized Taylor principle should apply to situations where $\phi_i < 0$. Economically, even if one believes that it is appropriate for a benevolent policy maker to choose a positive value for the interest rate response coefficient to inflation in the Taylor rule, one still cannot rule out the possibility that an incompetent or ill-informed policy maker might react differently. Moreover, Rotemberg and Woodford (1999a, Page 83) have shown that the *optimal* Taylor rule may involve a negative value of this parameter. In Section IX we exploit the fact that one or more regimes may be associated with a negative value for the inflation response coefficient to provide an example where there exist multiple bounded sunspot equilibria even when the generalized Taylor principle is satisfied.

Second, stationarity and boundedness are equivalent for the stochastic *linear* system (11) on which the generalized Taylor principle is based, because we assume that all the shocks are restricted to be bounded. But Condition 1 itself cannot distinguish stationary and bounded solutions and for the MSRE system (3) stationarity and boundedness are *not* equivalent. Consequently, the two different RE systems (3) and (11) cannot be equivalent.

VI. The MSV Solution

In a companion to this paper, Farmer Waggoner and Zha (2006) suggest MSV solutions as a way to make progress in the study of MSRE models. In this section, we define and find the MSV solution of (3). We also formally define the notion of mean squate stability.

Because (3) is a purely forward looking system, the MSV solution of (3) is the unique solution of the form

$$\pi_t = g_{\xi_t} r_t + h_{\xi_t} \varepsilon_t. \tag{13}$$

[5]See Farmer, Waggoner, and Zha (2007) for a proof of this result.

A solution of this kind may or may not exist. If we define

$$X = \begin{bmatrix} \phi_1 - \rho p_{1,1} & -\rho p_{2,1} \\ -\rho p_{1,2} & \phi_2 - \rho p_{2,2} \end{bmatrix}, \tag{14}$$

then the following proposition gives a necessary and sufficient condition for the existence of the MSV solution.

Proposition 1. There exists a unique solution of the the form of (13) if and only if the matrix X is invertible. In that case,

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = X^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } h_i = \frac{\kappa_i}{\phi_i}. \tag{15}$$

Proof. See Appendix A. □

In the light of this result, we assume that X is invertible and so the MSV solution will exist.

Following standard usage in the probability literature on Markov switching models (e.g., Costa, Fragoso, and Marques 2004), we define stationarity to be the existence of limiting first and second moments.

Definition 1 (Mean Square Stability). A stochastic process $\{x_t\}_{t=1}^{\infty}$ is mean square stable if there exist real numbers μ and φ such that

$$\lim_{s\to\infty} E_t \left[x_{t+s}\right] = \mu,$$
$$\lim_{s\to\infty} E_t \left[x_{t+s}^2\right] = \varphi.$$

Mean square stability is stronger than being bounded in mean and it is the appropriate stability concept if one wants to conduct statistical inference. It is widely used in the engineering literature and has been used in an economic application, among others, by Svensson and Williams (2005). An alternative to mean square stability is covariance stationarity in the sense of Hamilton (1994), or asymptotic covariance stationarity – a slightly weaker condition. Asymptotic covariance stationarity implies mean square stability, although the converse is not true in general. However, for the class of solutions studied in this paper, Theorem 3.33 of Costa, Fragoso, and Marques (2004) implies that these two notions are equivalent. Because of this fact, we will refer to solutions that satisfy the mean-square stability criterion as "stationary".

Proposition 2. If the Markov process $\{\xi_t\}_{t=1}^{\infty}$ is ergodic, then MSV solution (13) is stationary.

Proof. By assumption, the processes $\{\varepsilon_t\}_{t=1}^{\infty}$, $\{\nu_t\}_{t=1}^{\infty}$, and $\{\xi_t\}_{t=1}^{\infty}$ are independent and the processes $\{\varepsilon_t\}_{t=1}^{\infty}$ and $\{\nu_t\}_{t=1}^{\infty}$ are stationary. Since $\{\xi_t\}_{t=1}^{\infty}$ is ergodic, it will also be stationary. So, the processes $\{\varepsilon_t\}_{t=1}^{\infty}$, $\{r_t\}_{t=1}^{\infty}$, and $\{\xi_t\}_{t=1}^{\infty}$ will all be independent and stationary. The result now follows from Eq (13) and Definition 1. □

In light of this proposition, we assume that the Markov process $\{\xi_t\}_{t=1}^{\infty}$ is ergodic.

VII. A Class of Indeterminate Solutions

Since the existence of a unique determinate solution is often viewed as a desirable feature of a model (Rotemberg and Woodford 1999b, King 2000), we address the question: Is the MSV solution (13) unique in the class of all stationary solutions? Often the answer to this question is negative and we illustrate this point by first constructing a class of indeterminate solutions that are bounded in mean. However, only a subset of these will be stationary. This is very different from the constant parameter case, where for solutions of the type we construct these two notions are equivalent. These examples are instructive since they suggest that determinacy in MSRE models is much more delicate than in the constant parameter case.

Consider the change of variables defined by

$$w_t = \pi_t - \left(g_{\xi_t} r_t + \frac{\kappa_{\xi_t}}{\phi_{\xi_t}} \varepsilon_t \right). \tag{16}$$

Since $g_{\xi_t} r_t + \frac{\kappa_{\xi_t}}{\phi_{\xi_t}} \varepsilon_t$ is the MSV solution of (3), it is easy to see that π_t is a solution of (3) if and only if w_t is a solution of

$$w_t = \frac{1}{\phi_{\xi_t}} E_t \left[w_{t+1} \right]. \tag{17}$$

While it is easier to work directly with w_t, the reader should bear in mind that the behavior of the inflation variable π_t can be easily recovered from (16). Rearranging (17), one obtains

$$w_{t+1} = \phi_{\xi_t} w_t + \eta_{t+1}, \tag{18}$$

where η_{t+1} is the expectational error $w_{t+1} - E_t w_{t+1}$, which implies that $E_t \left[\eta_{t+1} \right] = 0$. On the other hand, if $\{\eta_t\}_{t=2}^{\infty}$ is any mean zero process and w_1 is any initial condition, then (18) defines a solution of (17).

Consider the expectational error of the form

$$\eta_{t+1} = \alpha_{\xi_{t+1}, \xi_t} \phi_{\xi_t} w_t + \beta_{\xi_{t+1}} \left(m \varepsilon_{t+1} + \gamma_{t+1} \right) \tag{19}$$

where the $\alpha_{i,j}$ must satisfy

$$p_{1,1} \alpha_{1,1} + p_{2,1} \alpha_{2,1} = p_{1,2} \alpha_{1,2} + p_{2,2} \alpha_{2,2} = 0 \tag{20}$$

and the stochastic process $\{\gamma_t\}_{t=2}^{\infty}$ has bounded support, is independently and identically distributed with zero mean, and is independent of $\{\xi_t\}_{t=1}^{\infty}$, $\{\varepsilon_t\}_{t=1}^{\infty}$, and $\{\nu_t\}_{t=1}^{\infty}$. Condition (20) and the independence of γ_{t+1} and ξ_{t+1} ensure that $E_t \eta_{t+1} = 0$. The solution $\{w_t\}_{t=1}^{\infty}$ of Eq (17) defined by Eqs (18), (19), and any initial condition w_1 will be of the form

$$w_{t+1} = \left(1 + \alpha_{\xi_{t+1}, \xi_t} \right) \phi_{\xi_t} w_t + \beta_{\xi_{t+1}} \left(m \varepsilon_{t+1} + \gamma_{t+1} \right). \tag{21}$$

Theorems 3.33 and 3.9 from Costa, Fragoso, and Marques (2004) give necessary and sufficient conditions for $\{w_t\}_{t=1}^{\infty}$ defined by Eq (21) to be stationary[6]. The conditions only involve the coefficients $(1+\alpha_{i,j})\,\phi_j$ and the transition probabilities $p_{i,j}$.

The solutions of Eq (3) given by Eqs (16) and (21) are exactly analogous to the solutions of Eq (5) given by Eqs (9) and (10) in the constant parameter case. There are, however, two important differences. The first is that the autoregressive parameter in Eq (21) involves the a new parameter $\alpha_{i,j}$, which is not the case for the autoregressive paramenter in Eq (10). So, unless a representation of $\alpha_{i,j}$ in terms of the other parameters can be found, we are faced with checking an infinite number of cases in order to determine uniqueness. The second is that the notions of stationarity and boundedness are both equivalent to the notion of bounded in mean for Eq (10), while this is not true for Eq (21). Because of this, the researcher must take a stand on which properties solutions are required to have.

In the remainder of this section, we apply these results to a specific example. First, we characterize the kind of central banker that acts in each regime. The idea is that the policy maker in regime 1 is an inflation hawk and the policy maker in regime 2, an inflation dove. This implies that

$$|\phi_1| > 1 > |\phi_2| > 0. \tag{22}$$

In line with the existing literature on the Taylor principle, we do not assume that the inflation response coefficients, ϕ_1 and ϕ_2, must be positive. As in Section III, the constant parameter model with $\phi_{\xi_t} = \phi_1$ has a unique stationary solution and the constant parameter model with $\phi_{\xi_t} = \phi_2$ has a continuum of stationary solutions.

Next, define

$$\alpha_{1,1} = -1,\ \alpha_{2,1} = \frac{p_{1,1}}{p_{2,1}},\ \alpha_{1,2} = -1,\ \alpha_{2,2} = \frac{p_{1,2}}{p_{2,2}}, \beta_1 = 0, \text{and } \beta_2 = 1. \tag{23}$$

Note that this implicitly implies that both $p_{2,1}$ and $p_{2,2}$ are non-zero, or equivalently, that the first state is not absorbing and the second state is not reflecting. The solution $\{w_t\}_{t=1}^{\infty}$ of Eq (17) generated by Eqs (18), (19), an initial condition w_1, and the parameters $\alpha_{i,j}$ and β_i given by (23) will have the form

$$w_{t+1} = \begin{cases} 0 & \text{if } \xi_t = 1 \text{ and } \xi_{t+1} = 1 \\ \frac{\phi_1}{p_{2,1}} w_t + (m\varepsilon_{t+1} + \gamma_{t+1}) & \text{if } \xi_t = 1 \text{ and } \xi_{t+1} = 2 \\ 0 & \text{if } \xi_t = 2 \text{ and } \xi_{t+1} = 1 \\ \frac{\phi_2}{p_{2,2}} w_t + (m\varepsilon_{t+1} + \gamma_{t+1}) & \text{if } \xi_t = 2 \text{ and } \xi_{t+1} = 2 \end{cases} \tag{24}$$

[6]The corresponding solution of Eq (3) is

$$\pi_{t+1} = \left(1+\alpha_{\xi_{t+1},\xi_t}\right)\phi_{\xi_t}\pi_t - \left(1+\alpha_{\xi_{t+1},\xi_t}\right)\phi_{\xi_t}\left(g_{\xi_t} r_t + \frac{\kappa_{\xi_t}}{\phi_{\xi_t}}\varepsilon_t\right) + g_{\xi_{t+1}} r_{t+1} + \frac{\kappa_{\xi_{t+1}}}{\phi_{\xi_{t+1}}}\varepsilon_{t+1} + \beta_{\xi_{t+1}}\left(m\varepsilon_{t+1} + \gamma_{t+1}\right).$$

The necessary and sufficient conditions for $\{\pi_t\}_{t=1}^{\infty}$ to be stationary are identical to those for $\{w_t\}_{t=1}^{\infty}$.

This solution has the property that is switchs between the MSV solution in the first state (the determinate state) and an autoregressive sunspot solution in the second state (the indeterminate state). The following proposition characterizes the long-run behavior of this solution.

Proposition 3. Let $\{w_t\}_{t=1}^{\infty}$ be the stochastic process generated by (24) and any initial condition w_1.

(1) $\{w_t\}_{t=1}^{\infty}$ is bounded in mean.
(2) $\{w_t\}_{t=1}^{\infty}$ is stationary if and only if $\left|\frac{\phi_1^2}{p_{2,2}}\right| < 1$.

Proof. See Appendix B. □

Though we shall not prove it, it is also the case that $\{w_t\}_{t=1}^{\infty}$ is bounded if and only if $\left|\frac{\phi_1}{p_{2,2}}\right| < 1$. In the next section we further consider the properties of these solutions.

VIII. Stationary Solutions

In Sections VIII and IX we provide two examples of indeterminate solutions based on the work in the previous section. The first example satisfies the mean-square stability condition so that the solution is stationary. The second example satisfies the more restricted Davig-Leeper definition of a stationary equilibrium for which the stationary invariant distribution is bounded. In both examples, we show that the generalized Taylor principle fails to hold.

Consider the process $\{w_t\}_{t=1}^{\infty}$ generated by (24) and any initial condition w_1. Notice, by Proposition 3, that the stationarity of our constructed sunspot equilibrium depends only on the values of the parameters in the second regime and on the probability of staying in that regime. Notably, it does not depend on the probability of staying in the first (determinate) regime nor does it depend on the degree to which the inflation hawk (the policy maker in the determinate regime) follows an active policy.

Our first example, illustrated in Figure 1, displays the MSV solution in the first panel and the paths of inflation associated with two different sunspot equilibria in the second and third panels. These two sunspot equilibria are associated with two different parameterizations of the economy.

For both parameterizations we set $\rho = 0.9$, $\phi_1 = 2.2$, $p_{1,1} = 0.98$, and $p_{2,2} = 0.995$. These parameter values correspond to monetary policy that would generate a unique equilibrium if the first regime were treated in isolation. The parameterizations differ by allowing alternative values of the interest rate coefficient in regime 2. In one case we set $\phi_2 = 0.9951$ and in the other $\phi_2 = 0.9949$. In both cases monetary policy in the second regime would lead to indeterminacy if the regime were treated in isolation. With $\phi_2 = 0.9951$, however, the generalized Taylor principle of Davig and Leeper (2007) implies that the MSV solution would be the *only* stationary solution. With $\phi_2 = 0.9949$, their uniqueness condition as stated in Condition 1 is violated. However,

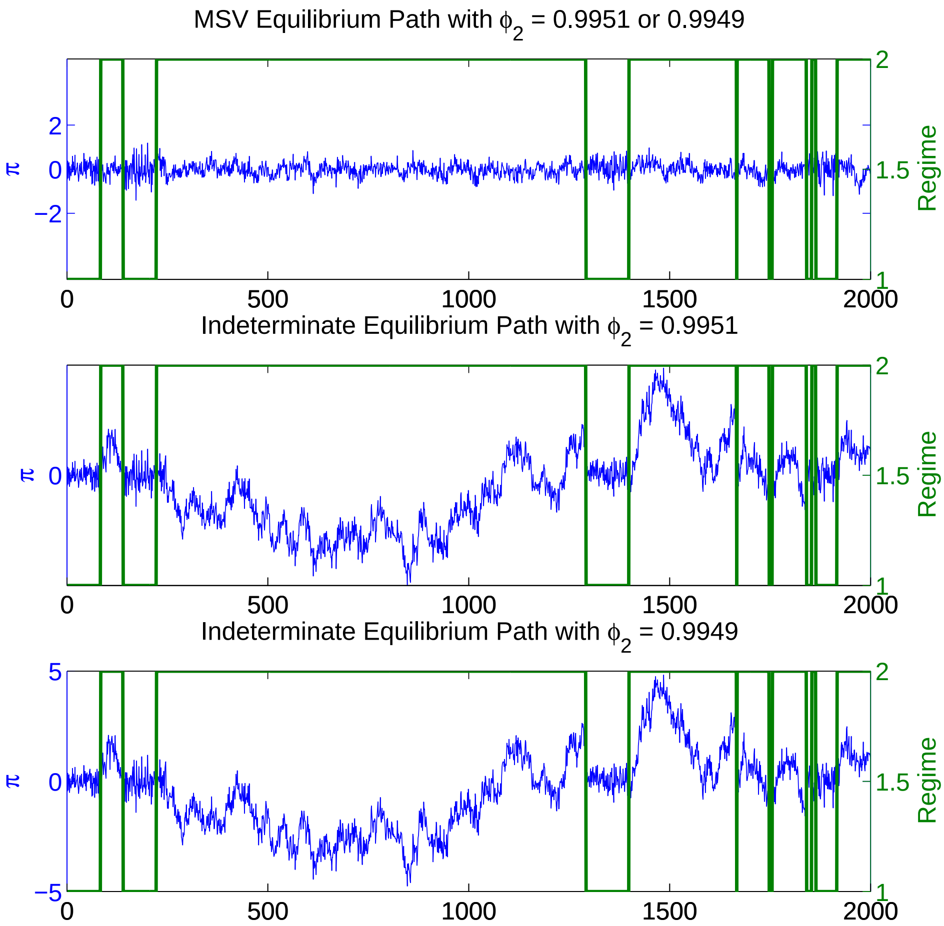

FIGURE 1. Simulated paths of inflation (deviations from the target) with $\phi_1 = 2.2, p_{1,1} = 0.98, p_{2,2} = 0.995$.

using the mean-square-stability criterion, the indeterminate solutions are stationary in both cases.[7]

It is worth noting that the paths associated with the policies $\phi_2 = 0.9951$ and $\phi_2 = 0.9949$ are hard to tell apart even though one violates the generalized Taylor principle and the other does not. In both cases, inflation in the indeterminate

[7]We parameterize this example by setting ε_t to be $N(0,1)$ and ν_t to be $N(0,0.01)$, both truncated at four standard deviations. The values of $\frac{\kappa_1}{\phi_1}$ and $\frac{\kappa_2}{\phi_2}$ are 0.4 and 0.2 so that the size of the fundamental shock in the first regime is twice the size in the second regime. The initial condition is $\pi_1 = g_1 r_1 + \frac{\kappa_1}{\phi_1}\varepsilon_1$, or equivalently $w_1 = 0$. All the paths are based on the same sequence of the real interest rate r_t and the same sequence of fundamental shocks ε_t. The non-fundamental shock $\gamma_t = 0$ and $m = \frac{\kappa_2}{\phi_2}$. Therefore, in the indeterminate equilibrium the size of the random shock is the same for both regimes. These assumptions allow us to isolate the effects of indeterminacy from the the effects of different shock variances across regimes.

equilibrium is much more persistent than that in the MSV equilibrium. This is consistent with Clarida, Galí, and Gertler's (2000a) interpretation of the U.S. data. They attribute the serially-dependent and volatile behavior of inflation in the 1970s to indeterminate monetary policy and the remarkable reduction in volatility and serial dependence of the Volcker-Greenspan years to the successful application of a Taylor rule that implements an MSV equilibrium. In both examples presented in Figure 1, the stochastic process that describes equilibrium is stable around the steady state. This is precisely our definition of stationarity for MSRE models.

IX. Bounded Solutions

In this section we discuss probability-one bounded solutions with bounded inputs (shocks). For analytical clarity we set the real interest rate r_t to zero and $\kappa_{\xi_t} = 1$ in (1) and consider the following model[8]

$$\phi_{\xi_t}\pi_t = E_t\pi_{t+1} + \varepsilon_t. \tag{25}$$

For this model, there exists an MSV solution

$$\pi_t = \frac{\varepsilon_t}{\phi_{\xi_t}}. \tag{26}$$

Under what conditions, is this the unique *bounded* solution? The answer is not as simple as that for the fixed-regime case studied in Section III. Consider the example where we set $\phi_1 = 24/9$ and $\phi_2 = -1/3$ with $p_{11} = 0.01$ and $p_{22} = 0.5$. The policy maker in the first regime is an inflation hawk, while monetary policy in the second regime would lead to indeterminacy if this regime were absorbing. Since we assume that the structural shocks $\{\varepsilon_t\}$ are bounded, the indeterminate solution represented by Eq (24) will be bounded as well. Hence, this example satisfies the uniqueness criterion of Davig and Leeper (2007). Our example demonstrates that a positive value of ϕ_i in both regimes is essential for this result to hold. This ad hoc positivity restriction is *not* innocuous because Condition 1 holds no matter whether ϕ_i is negative or not and because it rules out policies in which the coefficient of the Taylor rule is negative.

As argued before, under the expanded linear system (11) with bounded inputs, stationarity and boundedness are equivalent concepts. Since there is no equivalence between the expanded *linear* RE system (11) and the original MSRE model (3), we work directly on the original nonlinear system and provide a general uniqueness condition as follows.

Proposition 4. If the ϕ_1 and ϕ_2 are non-zero, the Markov-switching model represented by Eq (3) has multiple bounded solutions if and only if at least one of the eigenvalues of

$$\begin{bmatrix} |\phi_1|^{-1} & 0 \\ 0 & |\phi_2|^{-1} \end{bmatrix} \begin{bmatrix} p_{1,1} & p_{2,1} \\ p_{1,2} & p_{2,2} \end{bmatrix}$$

[8]Conditions for indeterminacy and determinacy for this model do not depend on the serial correlation of r_t and the value of κ_{ξ_t}.

is greater than or equal to one in absolute value.

Proof. See Appendix C. □

One can easily show that the condition in this proposition is the same as Condition 1 if and only if ϕ_i is restricted to be positive a priori. Since the uniqueness condition for the expanded *linear* system (11) on which the generalized Taylor principle is built *is* given by Condition 1, it follows that the two systems are *not* equivalent. It can also be shown that these two systems are different even for the case of $\phi_i > 0$, although in this case the uniqueness conditions for bounded equilibria *do* turn out to be the same.

X. Conclusion

We have shown in this paper that the distinction between linear and MSRE models is important but its consequences for equilibrium are not well understood. We have demonstrated that the properties of uniqueness, stationarity, and boundedness, even for the simple MSRE model of monetary policy studied in this paper, are fundamentally different from those of linear models.

For the one-equation example we have provided a necessary and sufficient condition for the uniqueness of bounded equilibria in the MSRE model. Characterizing the full class of equilibria in a general MSRE model remains a challenging task. We believe that progress can be made by adopting the MSV equilibrium concept advocated by McCallum (1983), as shown in a companion paper by Farmer, Waggoner, and Zha (2006).

Appendix A. Proof of Proposition 1

Substituting Eq (13) into Eq (3) we have,

$$\begin{aligned}\phi_{\xi_t}\left(g_{\xi_t} r_t + h_{\xi_t}\varepsilon_t\right) &= E_t\left[g_{\xi_{t+1}} r_{t+1} + h_{\xi_{t+1}}\varepsilon_{t+1}\right] + r_t + \kappa_{\xi_t}\varepsilon_t \\ &= \left(p_{1,\xi_t} g_1 + p_{2,\xi_t} g_2\right)\rho r_t + r_t + \kappa_{\xi_t}\varepsilon_t\end{aligned}$$

Gathering like terms we see that

$$\left(\phi_{\xi_t} h_{\xi_t} - \kappa_{\xi_t}\right)\varepsilon_t = 0$$

$$\left(\phi_{\xi_t} g_{\xi_t} - \left(p_{1,\xi_t} g_1 + p_{2,\xi_t} g_2\right)\rho - 1\right) r_t = 0$$

The first equation implies that $h_{\xi_t} = \kappa_{\xi_t}/\phi_{\xi_t}$ and the second equation can be written in matrix form as

$$X\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where X is defined by Eq (14). This equation will have a unique solution if and only if the matrix X is invertible. This completes the proof of Proposition 1.

Appendix B. Proof of Proposition 3

To prove Proposition 3, we apply the results of Costa, Fragoso, and Marques (2004). They consider systems of the form

$$x_{t+1} = \Gamma_{\theta_t} x_t + G_{\theta_t}\omega_t.$$

If we define

$$x_t = w_{t-1}, \theta_t = \left(\xi_t, \xi_{t-1}\right), \Gamma_{\theta_t} = \left(1 + \alpha_{\xi_t,\xi_{t-1}}\right)\phi_{\xi_t}, G_{\theta_t} = \beta_{\xi_t}\begin{bmatrix} m & 1\end{bmatrix} \text{ and } \omega_t = \begin{bmatrix}\varepsilon_t \\ \gamma_t\end{bmatrix},$$

then Eq (21) is of the form required by Costa, Fragoso, and Marques 2004. Note that the transition matrix of the Markov process $\{\theta_t\}_{t=1}^{\infty}$ is

$$P_\theta = \begin{bmatrix} p_{1,1} & p_{1,1} & 0 & 0 \\ 0 & 0 & p_{1,2} & p_{1,2} \\ p_{2,1} & p_{2,1} & 0 & 0 \\ 0 & 0 & p_{2,2} & p_{2,2}\end{bmatrix}.$$

It follows from Theorem 3.35 and Theorem 2.9 that $\{w_t\}_{t=1}^{\infty}$ is bounded in mean if all the eigenvalues of $P'_\theta diag\left(\Gamma_i\right)$ are less than one in absolute value. For the $\alpha_{i,j}$ defined by (23), it is easy to see that this matrix has eigenvalues equal to 0 and ϕ_2, both of which are less than one in absolute value. Thus $\{w_t\}_{t=1}^{\infty}$ is bounded in mean.

It follows from Theorem 3.33 and Theorem 3.9 that $\{w_t\}_{t=1}^{\infty}$ is stationary if and only if all the eigenvalues of $P'_\theta diag\left(\Gamma_i^2\right)$ are less than one in absolute value. For the $\alpha_{i,j}$ defined by (23), it is easy to see that this matrix has eigenvalues equal to 0 and $\phi_2^2/p_{2,2}$. Thus $\{w_t\}_{t=1}^{\infty}$ is stationary if and only $\phi_2^2/p_{2,2}$ is less than one in absolute value.

Appendix C. Proof of Proposition 4

To prove Proposition 4, it suffices to consider the existence of non-zero solutions of the system

$$\phi_{\xi_t}\pi_t = E_t\pi_{t+1}. \tag{A1}$$

First we assume that (λ, u) is an eigenvalue-eigenvector pair of $diag\left(|\phi_i|^{-1}\right)P'$ with $|\lambda| \geq 1$. Define

$$\pi_t = \lambda^{-t}u_{\xi_t}$$

where u_i is the i^{th} coordinate of u. Since u is an eigenvector, at least one of the u_i is non-zero and so π_t is not zero for all t. Since $|\lambda| \geq 1$, π_t will be bounded. We show that π_t is a solution of Eq (A1). Because $diag(\phi_i)u = \lambda^{-1}P'u$,

$$\phi_{\xi_t}\pi_t = \lambda^{-t}\phi_{\xi_t}u_{\xi_t} = \lambda^{-(t+1)}\left(p_{1,\xi_t}u_1 + p_{2,\xi_t}u_2\right) = E_t\pi_{t+1}.$$

So, π_t is a non-zero bounded solution of Eq (A1). If either λ or u were complex, then π_t would be complex. However, both the real and imaginary components of π_t would be real solutions of Eq (A1) and at least one of these would be non-zero.

We now assume that there is a non-zero solution π_t of Eq (A1). We show that there is an eigenvalue of $diag\,|\phi_i|\,P'$ that is greater than or equal to one in absolute value. Define $v_i = sup_{\xi_t=i}\{\pi_t\}$ and let v be the vector consisting of the v_i. Because π_t is bounded, v_i is finite. Because π_t is a solution of Eq (A1), if $\xi_t = j$, then

$$\begin{aligned}
|\phi_j\pi_t| &= |E_t\pi_{t+1}| \\
&\leq E_t|\pi_{t+1}| = p_{1,j}E_t\left[|\pi_{t+1}|\,|\xi_{t+1}=1\right] + p_{2,j}E_t\left[|\pi_{t+1}|\,|\xi_{t+1}=2\right] \\
&\leq p_{1,j}v_1 + p_{2,j}v_2.
\end{aligned}$$

This implies that $|\phi_j|\,v_j \leq p_{1,j}v_1 + p_{2,j}v_2$ or $diag\left(|\phi_i|\right)v \leq P'v$. This is equivalent to there existing numbers c_i with $0 \leq c_i \leq 1$ such that

$$v = diag\left(c_i\,|\phi_i|^{-1}\right)P'v.$$

Because the elements of v are non-negative, as are the elements of the matrix $diag\left((1-c_i)\,|\phi_i|^{-1}\right)P'$, the elements of $v_1 = diag\left((1-c_i)\,|\phi_i|^{-1}\right)P'v$ will also be non-negative. Iterating the relation

$$\begin{aligned}
diag\left(|\phi_i|^{-1}\right)P'v &= diag\left(c_i\,|\phi_i|^{-1}\right)P'v + diag\left((1-c_i)\,|\phi_i|^{-1}\right)P'v \\
&= v + v_1,
\end{aligned}$$

gives

$$\left(diag\left(|\phi_i|^{-1}\right)P'\right)^k v = v + v_k$$

where the elements of v_k are non-negative. If all of the eigenvalues of $diag\left(|\phi_i|^{-1}\right)P'$ were strictly less than one, then

$$\lim_{k\to\infty}\left(diag\left(|\phi_i|^{-1}\right)P'\right)^k v = 0,$$

which is a contradiction. Thus, there must be at least one eigenvalue of $diag\left(|\phi_i|^{-1}\right)P'$ that is greater than or equal to one.

References

Blanchard, O. J., and C. M. Kahn (1980): "The Solution of Linear Difference Methods under Rational Expectations," *Econometrica*, 48, 1305–1313.

Boivin, J., and M. Giannoni (2006): "Has Monetary Policy Become More Effective?," *Review of Economics and Statistics*, 88(3), 445–462.

Cass, D., and K. Shell (1983): "Do Sunspots Matter?," *Journal of Political Economy*, 91, 193–227.

Clarida, R., J. Galí, and M. Gertler (2000a): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115(1), 147–180.

——— (2000b): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, CXV, 147–180.

Costa, O., M. Fragoso, and R. Marques (2004): *Discrete-Time Markov Jump Linear Systems*. Springer, New York.

Davig, T., and E. M. Leeper (2007): "Generalizing the Taylor Principle," *The American Economic Review*, 97(3), 607–635.

Farmer, R. E. A., D. F. Waggoner, and T. Zha (2006): "Minimal State Variable Solutions to Markov-Switching Rational Expectations Models," *UCLA mimeo.*

——— (2007): "Understanding the New-Keynesian Model when Monetary Policy Switches Regimes," *NBER Working Paper 12965*.

Hamilton, J. D. (1994): *Times series analsis*. Princeton University Press, Princeton, NJ.

King, R. G. (2000): "The New IS-LM Model: Language, Logic, and Limits," *Federal Reserve Bank of Richmond Economic Quarterly*, 86/3, 45–103.

King, R. G., and M. W. Watson (1998): "The Solution of Singular Linear Difference Systems Under Rational Expectations," *International Economic Review*, 39(4), 1015–1026.

Lubik, T. A., and F. Schorfheide (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *The American Economic Review*, 94(1), 190–219.

McCallum, B. T. (1983): "On Non-Uniqueness in Rational Expectations Models: An Attempt at Perspective," *Journal of Monetary Economics*, 11, 139–168.

Rotemberg, J., and M. Woodford (1999a): *Monetary Policy Rules*chap. 2, pp. 57–126. NBER.

Rotemberg, J. J., and M. Woodford (1999b): "Interest Rate Rules in an Estimated Sticky Price Model," in *Monetary Policy Rules*, ed. by J. B. Taylor, pp. 57–119. University of Chicago Press, Chicago.

Sims, C. A. (2001): "Solving Linear Rational Expectations Models," *Journal of Computational Economics*, 20(1-2), 1–20.

——— (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1), 1–20.

Svensson, L., and N. Williams (2005): "Monetary Policy with Model Uncertainty: Distribution Forecast Targeting," *Princeton University Mimeo*.

Woodford, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press, Princeton, N.J.

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